vector is $\vec{r}\,'(t_0)$. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then by taking $\frac{d}{dt}$ of $F(\vec{r}(t)) = k$ gives

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

components of ∇

components of $\vec{r}\,'(t)$

$\Rightarrow \nabla F \cdot \vec{r}\,'(t) = 0$, i.e., ∇F is perpendicular to every tangent vector on S, i.e., ∇F is perpendicular to S.

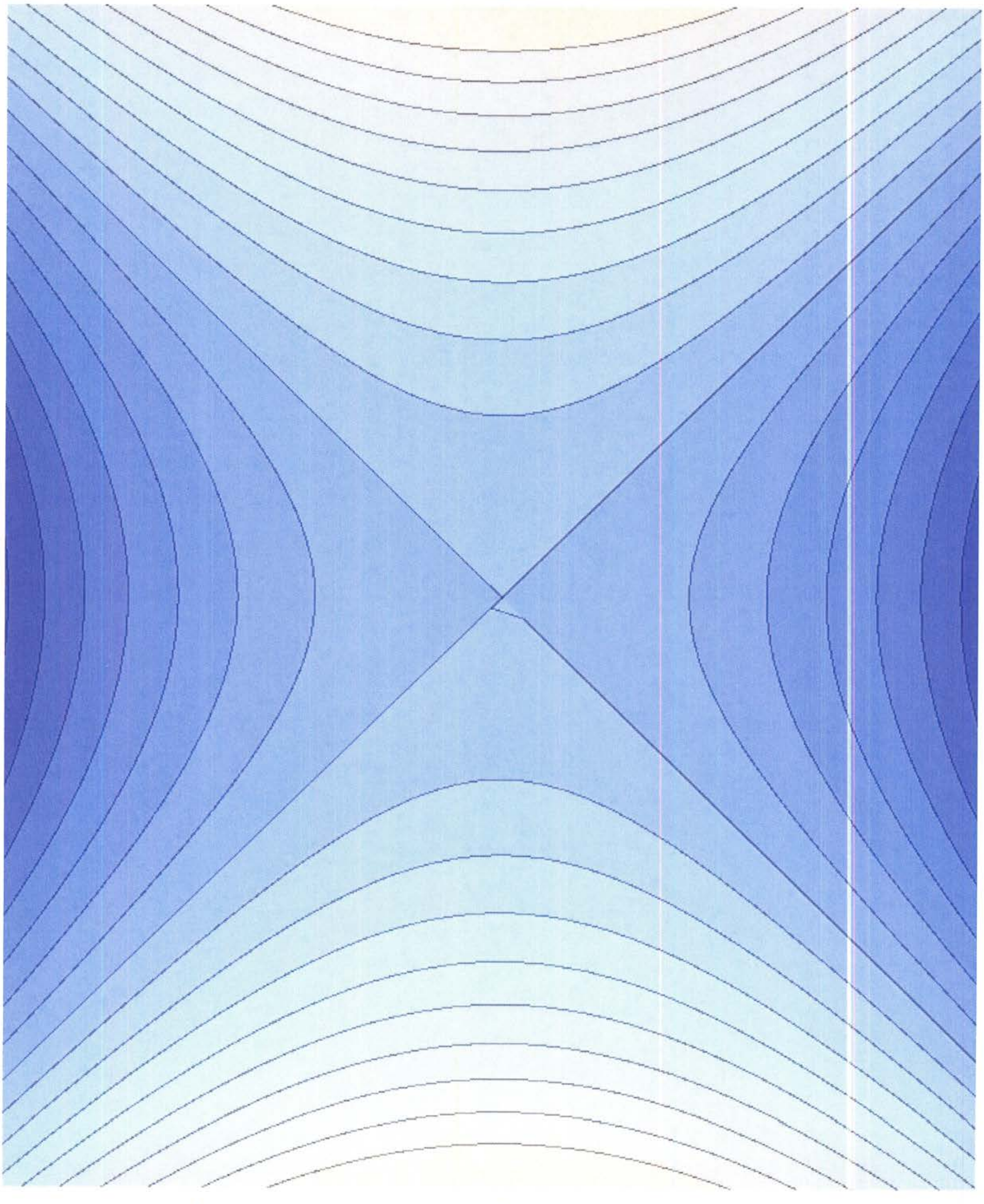
This, of course, applies to show ∇G is perpendicular to level curves of $G(x,y)$.

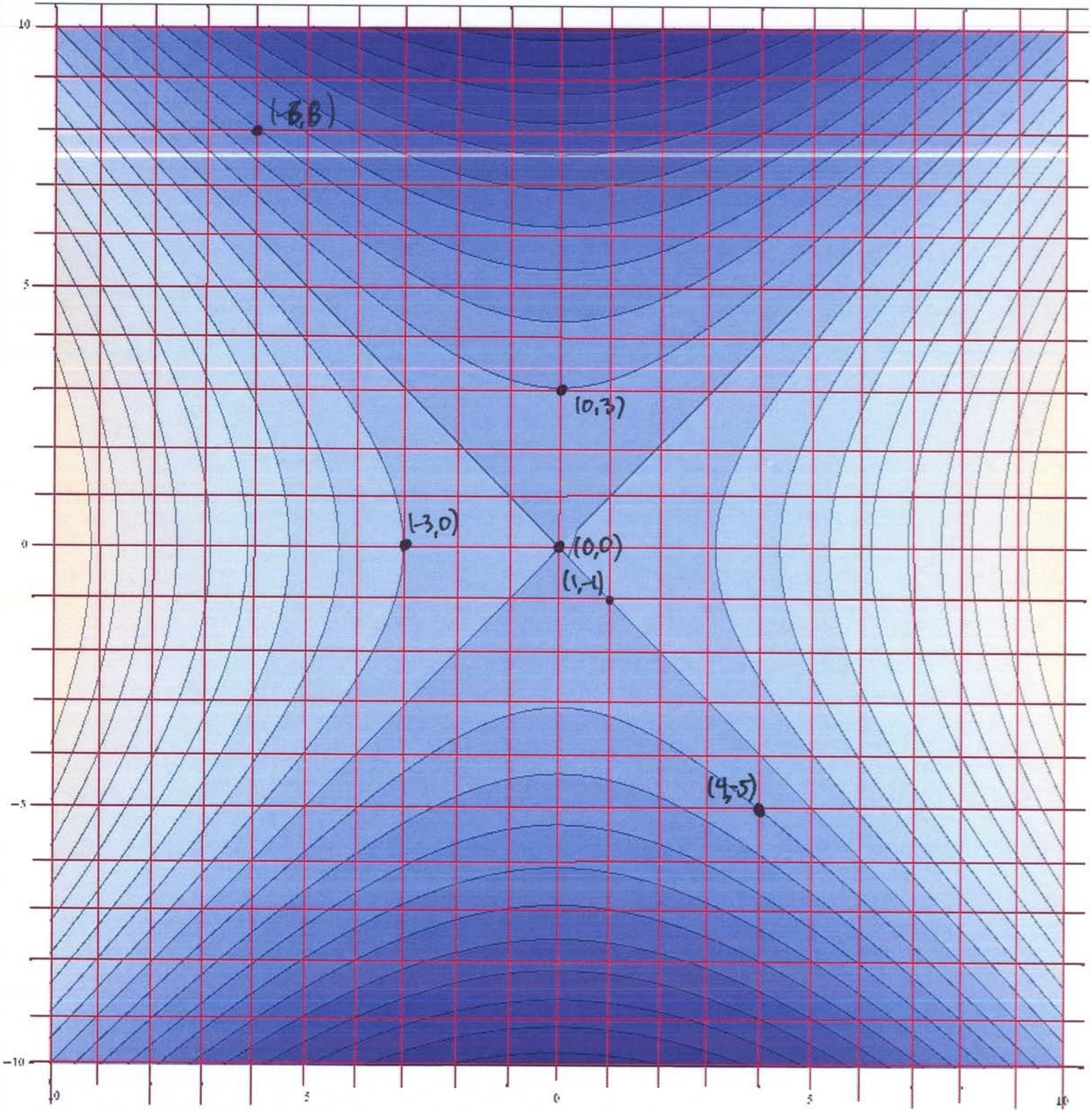
Lecture 10

Ex: Estimate the gradient at the indicated points given a contor plot of $f = f(x,y)$. (See next page.)

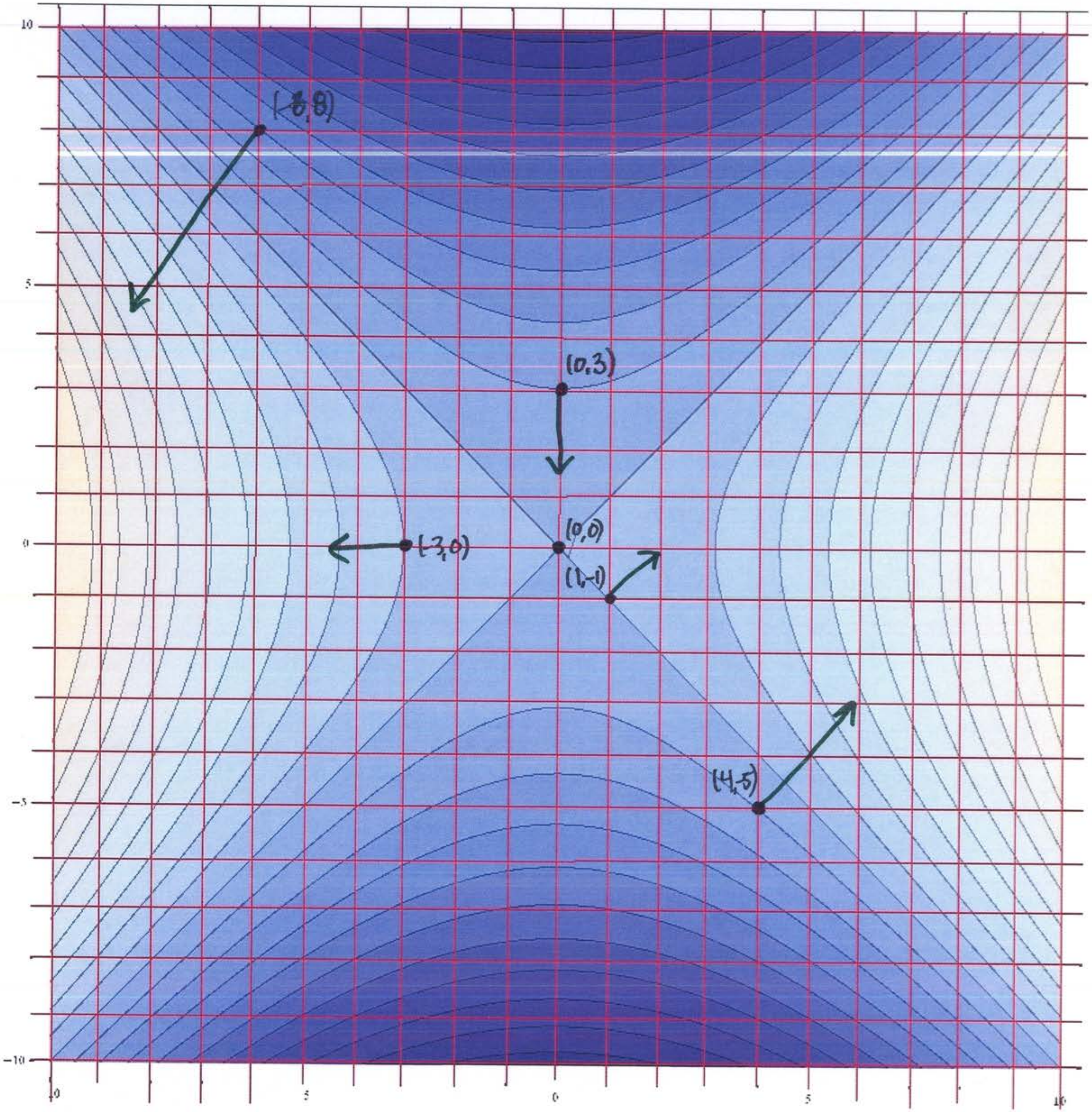
Now, since ∇F is perpendicular to level surfaces of F, we can use it to find tangent planes and normal lines to the surface.

10-2





(-8,8)
(0,3)
(-3,0)
(0,0)
(1,-1)
(4,-5)
10
5
0
-5
-10
5
0
5



Tan: increasing

Blue: decreasing

Ex: Find the tangent plane and normal line to the surface $y=x^2-z^2$ at $(4,7,3)$.

Sol: First, notice $y=x^2-z^2$ is a level surface of $F(x,y,z)=x^2-y-z^2$, namely $F(x,y,z)=0$.

$\nabla F = \langle 2x,-1,-2z\rangle$, $\nabla F(4,7,3)=\langle 8,-1,-6\rangle$.

The tangent plane has normal vector $\nabla F(4,7,3)$ and contains $(4,7,3)$, thus an equation is:

$$\langle 8,-1,-6\rangle \cdot \langle x-4, y-7, z-3\rangle = 0$$

-or-

$$8x-y-6z=7$$

The normal line points in the direction of $\nabla F(4,7,3)$ and contains $(4,7,3)$, thus an equation is:

$$\vec{l}(t) = \langle 4,7,3\rangle + t\langle 8,-1,-6\rangle = \langle 4+8t, 7-t, 3-6t\rangle.$$

◇

Another use:

Ex: Show that the ellipsoid $3x^2+2y^2+z^2=9$ is tangent to the sphere $x^2+y^2+z^2-8x-6y-8z+24=0$ at $(1,1,2)$.

Sol: Ellipsoid is level surface of: $F(x,y,z)=3x^2+2y^2+z^2$

Sphere level surface of: $G(x,y,z)=x^2+y^2+z^2-8x-6y-8z+24$

$\nabla G = \langle 2x-8, 2y-6, 2z-8 \rangle$, $\nabla G(1,1,2) = \langle -6,-4,-4 \rangle$

Since $\nabla F(1,1,2) = -\nabla G(1,1,2)$, the surfaces are tangent at $(1,1,2)$.

$\diamond$

One more use:

<u>Ex</u>: Find an equation for the tangent line to the intersection of the hyperboloid $x^2-y^2+z^2=6$ and the sphere $x^2+y^2+z^2=14$ at the point $(1,2,3)$.

<u>Sol</u>: Hyperboloid: level surface of $F(x,y,z) = x^2-y^2+z^2$

sphere : " " " $G(x,y,z) = x^2+y^2+z^2$

Normal vector to hyperboloid at $(1,2,3)$: $\nabla F(1,2,3) = \langle 2,-4,6 \rangle$

" " " sphere " " : $\nabla G(1,2,3) = \langle 2,4,6 \rangle$

Since $\nabla F(1,2,3)$ & $\nabla G(1,2,3)$ are both perpendicular to the curve of intersection, a vector tangent to it is

$$\nabla F(1,2,3) \times \nabla G(1,2,3) = \langle -48, 0, 16 \rangle$$

So, the tangent line is represented by:

$$\vec{l}(t) = \langle 1,2,3 \rangle + t\langle -48,0,16 \rangle = \langle 1-48t, 2, 3+16t \rangle$$

$\diamond$

14.7 - Maxima and Minima

Def: Let $f = f(x,y)$. A point (a,b) is called a

- local minimum if $f(a,b) \le f(x,y)$ for all (x,y) near (a,b). $f(a,b)$ is a local minimum value of f.
- local maximum if $f(a,b) \ge f(x,y)$ for all (x,y) near (a,b). $f(a,b)$ is a local maximum value of f.

A local minimum/maximum is called an absolute minimum/maximum if the respective inequalities hold for all (x,y) in the domain of f.

The procedure for finding these mirrors that in Calc I:

- find the critical points (a,b) of f, i.e., points (a,b) such that $\nabla f(a,b) = \vec{0}$.
- compute the determinant of the Hessian of f at (a,b): The Hessian is

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix},$$

so the number we want is:

$$D(a,b) = \det Hf(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

- compute $f_{xx}(a,b)$
- classify using:

Second Derivatives Test:

Suppose that f has continuous second partials near a point (a,b) such that $\nabla f(a,b)=\vec{0}$, i.e., (a,b) is a critical point of f. Let $D(a,b)=\det Hf(a,b)$, then:

- if $D(a,b)>0$ & $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum value of f.
- if $D(a,b)>0$ & $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum value of f.
- if $D(a,b)<0$, then (a,b) is a saddle point
- if $D(a,b)=0$, the test fails.

Ex: Find and classify the critical points of

$$f(x,y)=y^3+3x^2y-6x^2-6y^2+2.$$

Sol: First: $\nabla f=\langle 6xy-12x,\ 3y^2+3x^2-12y\rangle$

$$\nabla f=\vec{0}= \begin{cases} 6xy-12x=0 & (1) \\ 3y^2+3x^2-12y=0 & (2) \end{cases}$$

$(1) \Rightarrow 6x(y-2)=0 \Rightarrow x=0$ or $y=2$

$\boxed{x=0}$ $(2) \Rightarrow 3y^2-12y=0 \Rightarrow 3y(y-4)=0 \Rightarrow y=0$ or $y=4$

So, two critical points are $(0,0)$ & $(0,4)$.

$\boxed{y=2}$ $(2) \Rightarrow 12+3x^2-24=0 \Rightarrow 3x^2=12 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

Two more critical points are: $(-2,2)$ & $(2,2)$.

Now: $f_{xx} = 6y - 12$ &

$$Hf = \begin{pmatrix} 6y-12 & 6x \\ 6x & 6y-12 \end{pmatrix}$$

So $D = (6y-12)^2 - 36x^2$

Now, we classify:

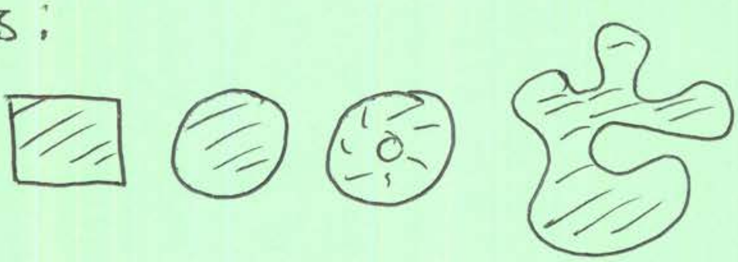
Crit. Pt	f_{xx}	D	type
$(0,0)$	-12	144	local max
$(0,4)$	12	144	local min
$(2,2)$	0	-144	saddle
$(-2,2)$	0	-144	saddle

$\diamond$

We can also ask about extremum when we restrict the domain of f. First some definitions:

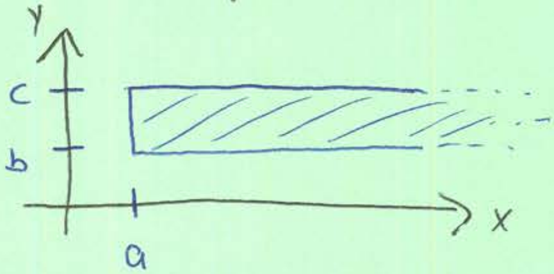
<u>Closed sets</u> are sets which contain all of their boundary points:

<u>Ex</u>:



<u>Non-ex</u>:

A set is <u>bounded</u> in $\mathbb{R}^2$ if it can fit in a disk. All of the above examples are bounded. An unbounded example is:



$$\{(x,y) \mid x \geq a,\ b \leq y \leq c\}$$

Lecture 11

Procedure for finding extrema on closed & bounded sets

This is the analog to finding extrema of $y=f(x)$ on $[a,b]$.

To find the absolute maximum and minimum values of a continuous function $f=f(x,y)$ on a closed & bounded set D:

① Find the values of f @ critical points of f <u>inside</u> D.

② Find the extreme values of f on the boundary.

③ The largest value from ① & ② is the absolute max & the smallest is the absolute min.

<u>Ex</u>: Find the absolute maximum & minimum values of $f(x,y)=x+y-xy$ on the closed & bounded set D which is the closed triangle with vertices $(0,0)$, $(0,2)$, and $(4,0)$.